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TIME DIFFERENTIATION OF TENSORS DEFINED ON A SURFACE MOVING THROUGH A THREE-DIMENSIONAL EUCLIDEAN SPACE*

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The well-known formulas for the derivatives of Eulerian and Lagrangian basis vectors are used to derive expressions for the derivatives of the surface, volume and double tensors defined on a surface moving through an Euclidean space. In the case of a plane moving through space with constant velocity, the results obtained correspond to the two-dimensional analogs of the results obtained in /1/. A relation connecting the derivatives in question with the derivative $(\delta/\delta t)$ is given, and the concept of the derivative $(\delta/\delta t)$ is introduced for the three-dimensional case.

In /1/ the author developed a theory of the time differentiation of tensors in the three-dimensional case, based on introducing Euclidean and Lagrangian basis vectors and a polyadic representation of tensors in these bases. The problem of the time differentiation of tensors was also considered in /2, 3/ using a general formulation, where a detailed analysis of the earlier work was also given. In /4—6/, in the course of studying wave propagation in continuous media, the derivative $(\delta/\delta t)$ of the components of three-dimensional vectors defined on a surface moving through a three-dimensional Euclidean space (at the wave front) was introduced. The results were generalized in /7/ to the case of surface and dual tensors defined on a moving surface.

1. The law of motion of the points belonging to a three-dimensional continuum is described by the equations

$$x^i = x^i(\xi^1, \xi^2, \xi^3, t), \quad \xi^k = \xi^k(x^1, x^2, x^3, t) \tag{1.1}$$

where x^i are the spatial (Eulerian) coordinates, ξ^k are the material (Lagrangian) coordinates and t is the time. The partial derivatives of the radius vector of the points of the space

$$E_i = \frac{\partial \mathbf{r}}{\partial x^i}, \quad E_i^\wedge = \frac{\partial \mathbf{r}}{\partial \xi^i} \tag{1.2}$$

define, respectively, the fixed Eulerian and moving Lagrangian basis. The tensor $\mathbf{T}$ with a typical distribution of the indices can be represented in invariant form /1/ as

$$\mathbf{T} = T^k_{\cdot m} E_k E^m = T^{\wedge k}_{\cdot m} E_k^\wedge E^{\wedge m} \tag{1.3}$$

The velocity vector of a particle with material coordinates is given by

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_\xi = v^i E_i = v^{\wedge i} E_i^\wedge; \quad v^i = \left(\frac{\partial x^i}{\partial t}\right)_\xi \tag{1.4}$$

$$v^{\wedge i} = \frac{\partial \xi^i}{\partial x^k} v^k$$

The time derivative of the tensor T can be obtained after establishing the formulas for differentiation of the basis vectors

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$$\left(\frac{\partial E_k}{\partial t}\right)_x = 0,\ \left(\frac{\partial E^k}{\partial t}\right)_x = 0,\ \left(\frac{\partial E_k}{\partial t}\right)_\xi = v^m \Gamma^p_{mk} E_p \tag{1.5}$$
$$\left(\frac{\partial E^k}{\partial t}\right)_\xi = -v^m \Gamma^k_{mp} E^p,\ \left(\frac{\partial E_k^\wedge}{\partial t}\right)_x = \frac{\partial v^{\wedge p}}{\partial \xi^k} E_p^\wedge$$
$$\left(\frac{\partial E^{\wedge k}}{\partial t}\right)_x = -\frac{\partial v^{\wedge k}}{\partial \xi^p} E^{\wedge p},\ \left(\frac{\partial E_k^\wedge}{\partial t}\right)_\xi = \nabla_k^\wedge v^{\wedge m} E^\wedge$$
$$\left(\frac{\partial E^{\wedge k}}{\partial t}\right)_\xi = -\nabla_p^\wedge v^{\wedge k} E^{\wedge p}$$

The second and fourth formulas of (1.5) were derived in /1/ for the initial instant when the Lagrangian and Eulerian bases coincide. Using the relation

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial \xi^k}\right)\right]_\xi = \frac{\partial}{\partial \xi^k}\left[\left(\frac{\partial \mathbf{r}}{\partial t}\right)_\xi\right]$$

we can show that the formulas (1.5) remain valid at any instant of time. The symbol $\nabla_k^\wedge$ means that the covariant differentiation is carried out with help of the Christoffel symbols $\Gamma^{\wedge p}_{km}$. Thus we arrive at the following expressions for the derivatives of the tensor:

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_x = \left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x E_k E^m \tag{1.6}$$
$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_x = \left[\left(\frac{\partial T^{\wedge k}_{\cdot m}}{\partial t}\right)_x + T^{\wedge p}_{\cdot m}\frac{\partial v^{\wedge k}}{\partial \xi^p} - T^{\wedge k}_{\cdot p}\frac{\partial v^{\wedge p}}{\partial \xi^m}\right] E_k^\wedge E^{\wedge m}$$
$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_\xi = \left[\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_\xi + T^p_{\cdot m} v^q \Gamma^k_{qp} - T^k_{\cdot p} v^q \Gamma^p_{mq}\right] E_k E^m$$
$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_\xi = \left[\left(\frac{\partial T^{\wedge k}_{\cdot m}}{\partial t}\right)_\xi + T^{\wedge p}_{\cdot m}\nabla_p^\wedge v^{\wedge k} - T^{\wedge k}_{\cdot p}\nabla_m^\wedge v^{\wedge p}\right] E_k^\wedge E^{\wedge m}$$

The following relations can be shown to hold:

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_\xi = \left(\frac{\partial \mathbf{T}}{\partial t}\right)_x + \mathbf{v}\cdot\nabla\mathbf{T} \tag{1.7}$$

$$\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x E_k E^m = \left[\left(\frac{\partial T^{\wedge k}_{\cdot m}}{\partial t}\right)_\xi - LT^{\wedge k}_{\cdot m}\right] E_k^\wedge E^{\wedge m} \tag{1.8}$$

where $LT^{\wedge k}_{\cdot m}$ is a Lie derivative of the components $T^{\wedge k}_{\cdot m}$ [8]

$$LT^{\wedge k}_{\cdot m} = v^{\wedge p}\frac{\partial T^{\wedge k}_{\cdot m}}{\partial \xi^p} - T^{\wedge p}_{\cdot m}\frac{\partial v^{\wedge k}}{\partial \xi^p} + T^{\wedge k}_{\cdot p}\frac{\partial v^{\wedge p}}{\partial \xi^m} = v^{\wedge p}\nabla_p^\wedge T^{\wedge k}_{\cdot m} - T^{\wedge p}_{\cdot m}\nabla_p^\wedge v^{\wedge k} + T^{\wedge k}_{\cdot p}\nabla_m^\wedge v^{\wedge p} \tag{1.9}$$

Using the notation

$$\frac{\delta T^{\wedge k}_{\cdot m}}{\delta t} \equiv \left(\frac{\partial T^{\wedge k}_{\cdot m}}{\partial t}\right)_\xi - LT^{\wedge k}_{\cdot m} \tag{1.10}$$

we can write (1.8) in the form

$$\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x E_k E^m = \frac{\delta T^{\wedge k}_{\cdot m}}{\delta t} E_k^\wedge E^{\wedge m} \tag{1.11}$$

Formulas (1.5) enable us to calculate the derivative of the metric tensor components

$$\left(\frac{\partial g^\wedge_{km}}{\partial t}\right)_\xi = \nabla_k^\wedge v_m^\wedge + \nabla_m^\wedge v_k^\wedge \tag{1.12}$$

whose contraction yields the equation of continuity

$$\left(\frac{\partial \rho}{\partial t}\right)_\xi + \rho\nabla\cdot\mathbf{v} = 0 \tag{1.13}$$

2. We define the surface moving through the space with help of the following equations:

$$x^i = x^i(u^1, u^2, t),\quad x^i = x^i(\omega^1, \omega^2, t) \tag{2.1}$$

Here u^α are the Eulerian surface coordinates and ω^α the Lagrangian surface coordinates connected by the relations

$$u^\alpha = u^\alpha(\omega^1, \omega^2, t),\quad \omega^\beta = \omega^\beta(u^1, u^2, t) \tag{2.2}$$

As usually used in the theory of surfaces in three-dimensional space, the latin indices take the values 1, 2, 3, and the greek indices the values 1, 2 /9/. We note that the introduction of the Eulerian coordinates to the moving surfaces is not trivial. We shall say that a point is "fixed" on the surface if its velocity is directed along the normal to the surface /10, 11/. Thus the equation

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_u = v_{(n)}\mathbf{n}, \quad \left(\frac{\partial x^i}{\partial t}\right)_u = v_{(n)}n^i \tag{2.3}$$

does in fact define the Eulerian coordinates u^α on the surface, while

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_\omega = \mathbf{v} = v^i E_i, \quad \left(\frac{\partial x^i}{\partial t}\right)_\omega = v^i \tag{2.4}$$

represents the velocity of the particle with material coordinates ω^α. We stress that $(\partial \mathbf{r}/\partial t)_\omega$ and $(\partial \mathbf{r}/\partial t)_\xi$ are different quantities. From now on we shall always take $\mathbf{v}$ as $(\partial \mathbf{r}/\partial t)_\omega$. The velocity of the point in the "two-dimensional" world is given by the partial derivative

$$\left(\frac{\partial u^\alpha}{\partial t}\right)_\omega = v^\alpha \tag{2.5}$$

and the following relation holds:

$$\left(\frac{\partial x^i}{\partial t}\right)_\omega = \left(\frac{\partial x^i}{\partial t}\right)_u + \frac{\partial x^i}{\partial u^\alpha}\left(\frac{\partial u^\alpha}{\partial t}\right)_\omega$$

$$v^i = v_{(n)}n^i + x^i_\alpha v^\alpha, \quad x_\alpha{}^i \equiv \frac{\partial x^i}{\partial u^\alpha}$$

The partial derivatives

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha}, \quad \mathbf{a}_\alpha^\wedge = \frac{\partial \mathbf{r}}{\partial \omega^\alpha} \tag{2.6}$$

form the local Eulerian and Lagrangian bases on the surface. The surface tensor $\mathbf{T}$ with typical distribution of indices can be written in the form

$$\mathbf{T} = T^{\alpha}_{\cdot\beta}\mathbf{a}_\alpha\mathbf{a}^\beta = T^{\wedge\alpha}_{\cdot\beta}\mathbf{a}_\alpha{}^\wedge\mathbf{a}^{\wedge\beta} \tag{2.7}$$

The following basic formulas are used in the time differentiation of tensors:

$$\left(\frac{\partial \mathbf{a}_\alpha}{\partial t}\right)_u = -v_{(n)}b_\alpha{}^\beta\mathbf{a}_\beta + \frac{\partial v_{(n)}}{\partial u^\alpha}\mathbf{n} \tag{2.8}$$

$$\left(\frac{\partial \mathbf{a}^\beta}{\partial t}\right)_u = v_{(n)}b_\gamma{}^\beta\mathbf{a}^\gamma + a^{\beta\gamma}\frac{\partial v_{(n)}}{\partial u^\gamma}\mathbf{n}$$

$$\left(\frac{\partial \mathbf{n}}{\partial t}\right)_u = -\frac{\partial v_{(n)}}{\partial u^\alpha}\mathbf{a}^\alpha, \quad \left(\frac{\partial \mathbf{n}}{\partial t}\right)_\omega = -\left(v^\gamma b_{\alpha\gamma} + \frac{\partial v_{(n)}}{\partial u^\alpha}\right)\mathbf{a}^\alpha$$

$$\left(\frac{\partial \mathbf{a}_\alpha}{\partial t}\right)_\omega = (v^\gamma G^\beta_{\alpha\gamma} - v_{(n)}b_\alpha{}^\beta)\mathbf{a}_\beta + \left(v^\gamma b_{\alpha\gamma} + \frac{\partial v_{(n)}}{\partial u^\alpha}\right)\mathbf{n}$$

$$\left(\frac{\partial \mathbf{a}^\beta}{\partial t}\right)_\omega = -(v^\gamma G^\beta_{\gamma\lambda} - v_{(n)}b_\lambda{}^\beta)\mathbf{a}^\lambda + a^{\alpha\beta}\left(v^\gamma b_{\alpha\gamma} + \frac{\partial v_{(n)}}{\partial u^\alpha}\right)\mathbf{n}$$

$$\left(\frac{\partial \mathbf{a}_\alpha{}^\wedge}{\partial t}\right)_u = \left(\frac{\partial v^{\wedge\beta}}{\partial \omega^\alpha} - v_{(n)}b^{\wedge\beta}_\alpha\right)\mathbf{a}_\beta{}^\wedge + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\mathbf{n}$$

$$\left(\frac{\partial \mathbf{a}^{\wedge\beta}}{\partial t}\right)_u = -\left(\frac{\partial v^{\wedge\beta}}{\partial \omega^\gamma} - v_{(n)}b^{\wedge\beta}_\gamma\right)\mathbf{a}^{\wedge\gamma} + a^{\wedge\alpha\beta}\frac{\partial v_{(n)}}{\partial \omega^\alpha}\mathbf{n}$$

$$\left(\frac{\partial \mathbf{n}}{\partial t}\right)_u = -\frac{\partial v_{(n)}}{\partial \omega^\alpha}\mathbf{a}^{\wedge\alpha}, \quad \left(\frac{\partial \mathbf{n}}{\partial t}\right)_\omega = -\left(v^{\wedge\gamma}b^\wedge_{\alpha\gamma} + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\right)\mathbf{a}^{\wedge\alpha}$$

$$\left(\frac{\partial \mathbf{a}_\alpha{}^\wedge}{\partial t}\right)_\omega = (\nabla_\alpha{}^\wedge v^{\wedge\beta} - v_{(n)}b^{\wedge\beta}_\alpha)\mathbf{a}_\beta{}^\wedge + \left(v^{\wedge\beta}b^\wedge_{\alpha\beta} + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\right)n$$

$$\left(\frac{\partial \mathbf{a}^{\wedge\beta}}{\partial t}\right)_\omega = -(\nabla_\gamma{}^\wedge v^{\wedge\beta} - v_{(n)}b^{\wedge\beta}_\gamma)\mathbf{a}^{\wedge\gamma} + a^{\wedge\alpha\beta}\left(v^{\wedge\gamma}b^\wedge_{\alpha\gamma} + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\right)\mathbf{n}$$

Here $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the components of the first and second quadratic form of the surface, and $G^\gamma_{\alpha\beta}$ are two-dimensional Christoffel symbols. The last three formulas of (2.8) were given in /12/. Using (2.7) and (2.8) we obtain

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_u = \left[\left(\frac{\partial T^\alpha_{\cdot\beta}}{\partial t}\right)_u - T^\rho_{\cdot\beta}b_\rho{}^\alpha v_{(n)} + T^\alpha_{\cdot\rho}b_\beta{}^\rho v_{(n)}\right]\mathbf{a}_\alpha\mathbf{a}^\beta + T^\alpha_{\cdot\beta}\frac{\partial v_{(n)}}{\partial u^\alpha}\mathbf{n}\mathbf{a}^\beta + T^\alpha_{\cdot\beta}a^{\beta\gamma}\frac{\partial v_{(n)}}{\partial u^\gamma}\mathbf{a}_\alpha\mathbf{n} \tag{2.9}$$

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_u = \left[\left(\frac{\partial T^{\wedge\alpha}_{\cdot\beta}}{\partial t}\right)_u + T^{\wedge\rho}_{\cdot\beta}\left(\frac{\partial v^{\wedge\alpha}}{\partial \omega^\rho} - v_{(n)}b^{\wedge\alpha}_\rho\right) - T^{\wedge\alpha}_{\cdot\rho}\left(\frac{\partial v^{\wedge\rho}}{\partial \omega^\beta} - v_{(n)}b^{\wedge\rho}_\beta\right)\right]\mathbf{a}_\alpha{}^\wedge\mathbf{a}^{\wedge\beta} +$$

$$T_{\cdot\beta}^{\wedge\alpha}\frac{\partial v_{(n)}}{\partial\omega^{\alpha}}\mathbf{n}\mathbf{a}^{\wedge\beta}+T_{\cdot\beta}^{\wedge\alpha}a^{\wedge\beta\gamma}\frac{\partial v_{(n)}}{\partial\omega^{\gamma}}\mathbf{a}^{\wedge}{}_{\alpha}\mathbf{n}$$

$$\left(\frac{\partial\mathbf{T}}{\partial t}\right)_{\omega}=\left[\left(\frac{\partial T^{\alpha}_{\cdot\beta}}{\partial t}\right)_{\omega}+T^{\rho}_{\cdot\beta}\left(v^{\lambda}G^{\alpha}_{\lambda\rho}-v_{(n)}b_{\rho}{}^{\alpha}\right)-\right.$$
$$\left.T^{\alpha}_{\cdot\rho}\left(v^{\lambda}G^{\rho}_{\lambda\beta}-v_{(n)}b_{\beta}{}^{\rho}\right)\right]\mathbf{a}_{\alpha}\mathbf{a}^{\beta}+T^{\alpha}_{\cdot\beta}\left(v^{\gamma}b_{\alpha\gamma}+\frac{\partial v_{(n)}}{\partial u^{\alpha}}\right)\mathbf{n}\mathbf{a}^{\beta}-$$
$$T^{\alpha}_{\cdot\beta}a^{\beta\rho}\left(v^{\lambda}b_{\lambda\rho}+\frac{\partial v_{(n)}}{\partial u^{\rho}}\right)\mathbf{a}_{\alpha}\mathbf{n}$$

$$\left(\frac{\partial\mathbf{T}}{\partial t}\right)_{\omega}=\left[\left(\frac{\partial T_{\cdot\beta}^{\wedge\alpha}}{\partial t}\right)_{\omega}+T_{\cdot\beta}^{\wedge\rho}\left(\nabla_{\rho}{}^{\wedge}v^{\wedge\alpha}-v_{(n)}b_{\rho}^{\wedge\alpha}\right)-\right.$$
$$\left.T_{\cdot\rho}^{\wedge\alpha}\left(\nabla_{\beta}{}^{\wedge}v^{\wedge\rho}-v_{(n)}b_{\beta}^{\wedge\rho}\right)\right]\mathbf{a}_{\alpha}{}^{\wedge}\mathbf{a}^{\wedge\beta}+T_{\cdot\beta}^{\wedge\alpha}\left(v^{\wedge\rho}b^{\wedge}_{\alpha\rho}+\right.$$
$$\left.\frac{\partial v_{(n)}}{\partial\omega^{\alpha}}\right)\mathbf{n}\mathbf{a}^{\wedge\beta}+T_{\cdot\beta}^{\wedge\alpha}a^{\wedge\beta\rho}\left(v^{\wedge\lambda}b^{\wedge}_{\lambda\rho}+\frac{\partial v_{(n)}}{\partial\omega^{\alpha}}\right)\mathbf{a}_{\alpha}{}^{\wedge}\mathbf{n}$$

The two-dimensional analogs of the formulas (1.7), (1.8), (1.10), (1.11) (∇_{Σ} is the surface del operator) remain valid

$$\left(\frac{\partial\mathbf{T}}{\partial t}\right)_{\omega}=\left(\frac{\partial\mathbf{T}}{\partial t}\right)_{u}+\mathbf{v}\cdot\nabla_{\Sigma}\mathbf{T} \tag{2.10}$$
$$\left(\frac{\partial T^{\alpha}_{\cdot\beta}}{\partial t}\right)_{u}\mathbf{a}_{\alpha}\mathbf{a}^{\beta}=\left[\left(\frac{\partial T_{\cdot\beta}^{\wedge\alpha}}{\partial t}\right)_{\omega}-LT_{\cdot\beta}^{\wedge\alpha}\right]\mathbf{a}_{\alpha}{}^{\wedge}\mathbf{a}^{\wedge\beta}$$
$$\frac{\delta T_{\cdot\beta}^{\wedge\alpha}}{\delta t}=\left(\frac{\partial T_{\cdot\beta}^{\wedge\alpha}}{\partial t}\right)_{\omega}-LT_{\cdot\beta}^{\wedge\alpha}$$
$$\left(\frac{\partial T^{\alpha}_{\cdot\beta}}{\partial t}\right)_{u}\mathbf{a}_{\alpha}\mathbf{a}^{\beta}=\frac{\delta T_{\cdot\beta}^{\wedge\alpha}}{\delta t}\mathbf{a}_{\alpha}{}^{\wedge}\mathbf{a}^{\wedge\beta}$$

$$LT_{\cdot\beta}^{\wedge\alpha}=v^{\wedge\rho}\frac{\partial T_{\cdot\beta}^{\wedge\alpha}}{\partial\omega^{\rho}}-T_{\cdot\beta}^{\wedge\rho}\frac{\partial v^{\wedge\alpha}}{\partial\omega^{\rho}}+T_{\cdot\rho}^{\wedge\alpha}\frac{\partial v^{\wedge\rho}}{\partial\omega^{\beta}}=v^{\wedge\rho}\nabla_{\rho}{}^{\wedge}T_{\cdot\beta}^{\wedge\alpha}-T_{\cdot\beta}^{\wedge\rho}\nabla_{\rho}{}^{\wedge}v^{\wedge\alpha}+T_{\cdot\rho}^{\wedge\alpha}\nabla_{\beta}{}^{\wedge}v^{\wedge\rho} \tag{2.11}$$

The $(\delta/\delta t)$-derivative operation on the surface tensor components was first considered in /7/. The fourth formula of (2.10) provides the relation connecting the operation of differentiating the tensor components in the Eulerian basis with constant u^{α}, and the $(\delta/\delta t)$-differentiation of the tensor components in the Lagrangian basis. This also clarifies the reason for choosing the notation $(\delta/\delta t)$ in (1.10). Thus the concept of $(\delta/\delta t)$-derivative characterizes the variation in the tensor components in both the surface and spatial case. We note that the $(\delta/\delta t)$-derivative of the tensor components on the surface does not descibe fully the variation of the tensor with time, since the terms with normal components appearing in (2.9) are not included in the discussion. The following derivative of the metric tensor components is of interest:

$$\left(\frac{\partial a^{\wedge}_{\alpha\beta}}{\partial t}\right)_{\omega}=\nabla_{\alpha}{}^{\wedge}v_{\beta}{}^{\wedge}+\nabla_{\beta}{}^{\wedge}v_{\alpha}{}^{\wedge}-2v_{(n)}b^{\wedge}_{\alpha\beta} \tag{2.12}$$

Its convolution leads to the equation of continuity at the surface /10/

$$\left(\frac{\partial\rho_{\Sigma}}{\partial t}\right)_{\omega}+\rho_{\Sigma}\nabla_{\Sigma}\cdot\mathbf{v}=0,\qquad\left(\frac{\partial\rho_{\Sigma}}{\partial t}\right)_{\omega}+\rho_{\Sigma}\left(\nabla_{\alpha}{}^{\wedge}v^{\wedge\alpha}-v_{(n)}b_{\alpha}^{\wedge\alpha}\right)=0 \tag{2.13}$$

When $v_{(n)}=\mathrm{const}$, $b^{\wedge}_{\alpha\beta}=0$, i.e. when a plane moves through space with constant velocity, the results of Sect.2 reduce to those of Sect.1, the only difference being that the latin indices are replaced by greek indices.

3. Let us derive the formulas for the differentiation of three-dimensional vectors

$$\mathbf{T}=T^{k}_{\cdot m}E_{k}E^{m} \tag{3.1}$$

regarded on the surface as functions of (u^{α},t) or (ω^{α},t). The time derivatives of the basis vectors are

$$\left(\frac{\partial E_{k}}{\partial t}\right)_{u}=v_{(n)}n^{p}\Gamma^{q}_{pk}E_{q},\qquad\left(\frac{\partial E^{m}}{\partial t}\right)_{u}=-v_{(n)}n^{p}\Gamma^{m}_{pq}E^{q} \tag{3.2}$$
$$\left(\frac{\partial E_{k}}{\partial t}\right)_{\omega}=v^{p}\Gamma^{q}_{pk}E_{q},\qquad\left(\frac{\partial E^{m}}{\partial t}\right)_{\omega}=-v^{p}\Gamma^{m}_{pq}E^{q}$$

Consequently

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_u = \left[\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_u + v_{(n)} n^p T^q_{\cdot m}\Gamma^k_{pq} - v_{(n)} n^p T^k_{\cdot q}\Gamma^q_{pm}\right] E_k E^m \tag{3.3}$$

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_\omega = \left[\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_\omega + v^p T^q_{\cdot m}\Gamma^k_{pq} - v^p T^k_{\cdot q}\Gamma^q_{pm}\right] E_k E^m$$

Using the relations

$$\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_u = \left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x + v_{(n)} n^p \frac{\partial T^k_{\cdot m}}{\partial x^p}$$

$$\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_\omega = \left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x + v^p \frac{\partial T^k_{\cdot m}}{\partial x^p}$$

in place of the formulas (1.3), we obtain

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_u = \left[\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x + v_{(n)} n^p \nabla_p T^k_{\cdot m}\right] E_k E^m \equiv \frac{\delta T^k_{\cdot m}}{\delta t} E_k E^m \tag{3.4}$$

$$\left(\frac{\partial \mathbf{T}}{\partial t}\right)_\omega = \left[\left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_x + v^p \nabla_p T^k_{\cdot m}\right] E_k E^m$$

The first formula of (3.4) shows that the components of the time derivative of the spatial tensor for constant u^α at the surface, are equal to the $(\delta/\delta t)$ -derivative of the components of this tensor studied in /7/. Using (2.10) and (3.3), we can write the (3.3) $(\delta/\delta t)$-derivative as follows:

$$\frac{\delta T^k_{\cdot m}}{\delta t} = \left(\frac{\partial T^k_{\cdot m}}{\partial t}\right)_\omega - v^\beta \nabla_\beta T^k_{\cdot m} + v^p T^q_{\cdot m}\Gamma^k_{pq} - v^p T^k_{\cdot q}\Gamma^q_{pm}, \qquad \nabla_\beta T^k_{\cdot m} = x_\beta{}^p \nabla_p T^k_{\cdot m} \tag{3.5}$$

4. Knowing the derivatives (2.8) and (3.2), we can easily differentiate the dual tensors provided that we write them in invariant form, e.g.

$$\mathbf{W} = W^{\alpha\cdot k}_{\cdot\beta\cdot m}\mathbf{a}_\alpha \mathbf{a}^\beta E_k E^m = W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}\mathbf{a}_\alpha{}^\wedge \mathbf{a}^{\wedge\beta} E_k E^m \tag{4.1}$$

In describing the dependence of the dual tensor components on a point on the surface, two cases are possible /7/: relations (2.1) are only implied, i.e. $W^{\alpha\cdot k}_{\cdot\beta\cdot m} = f_1(x^k, u^\alpha, t)$, $W^{\alpha\cdot k}_{\cdot\beta\cdot m} = f_2(x^k, \omega^\beta, t)$, or x^k are already eliminated using (2.1), i.e. $W^{\alpha\cdot k}_{\cdot\beta\cdot m} = f_3(u^\alpha, t)$, $W^{\alpha\cdot k}_{\cdot\beta\cdot m} = f_4(\omega^\alpha, t)$. The derivatives $(\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}/\partial t)_{x,u}$ and $(\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}/\partial t)_{x,\omega}$, correspond to the first case, and $(\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}/\partial t)_u$ and $(\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}/\partial t)_\omega$ to the second case.

Thus we have four different representations of $(\partial W/\partial t)_u$ and four different representations of $(\partial W/\partial t)_\omega$. Their derivation is not difficult, but cumbersome, and is therefore not given here. The relations connecting the differentiation of the dual tensor components on the Eulerian basis with the $(\delta/\delta t)$-derivative of the components of the same tensor on the Lagrangian basis is given by the formulas *

$$\left[\left(\frac{\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}}{\partial t}\right)_{x,u} + v_{(n)} n^q \nabla_q W^{\alpha\cdot k}_{\cdot\beta\cdot m}\right]\mathbf{a}_\alpha \mathbf{a}^\beta E_k E^m = \frac{\delta W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}}{\delta t}\mathbf{a}_\alpha{}^\wedge \mathbf{a}^{\wedge\beta} E_k E^m \tag{4.2}$$

$$\frac{\delta W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}}{\delta t} \equiv \left(\frac{\partial W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}}{\partial t}\right)_{x,\omega} - L W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m} + v_{(n)} n^q \nabla_q W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m} \tag{4.3}$$

$$\frac{\delta W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}}{\delta t} = \left(\frac{\partial W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m}}{\partial t}\right)_\omega - v^{\wedge\gamma}\nabla_\gamma{}^\wedge W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot m} + W^{\wedge\rho\cdot k}_{\cdot\beta\cdot m}\nabla_\rho{}^\wedge v^{\wedge\alpha} - W^{\wedge\alpha\cdot k}_{\cdot\rho\cdot m}\nabla_\beta{}^\wedge v^{\wedge\rho} + v^q\left(W^{\wedge\alpha\cdot p}_{\cdot\beta\cdot m}\Gamma^k_{pq} - W^{\wedge\alpha\cdot k}_{\cdot\beta\cdot p}\Gamma^p_{qm}\right) \tag{4.4}$$

The Lie derivative is given by the first formula of (2.11) and involves the greek indices only, the covariant differentiation $\nabla_q W^{\alpha\cdot k}_{\cdot\beta\cdot m}$ involves the latin indices only and is carried out with help of the three-dimensional Christoffel symbols, and the covariant derivative $\nabla_\gamma W^{\alpha\cdot k}_{\cdot\beta\cdot m}$ has the following structure /9/:

$$\nabla_\gamma W^{\alpha\cdot k}_{\cdot\beta\cdot m} = \frac{\partial W^{\alpha\cdot k}_{\cdot\beta\cdot m}}{\partial u^\gamma} + W^{\rho\cdot k}_{\cdot\beta\cdot m} G^\alpha_{\gamma\rho} - W^{\alpha\cdot k}_{\cdot\rho\cdot m} G^\rho_{\gamma\beta} + x_\gamma{}^p W^{\alpha\cdot q}_{\cdot\beta\cdot m}\Gamma^k_{pq} - x_\gamma{}^p W^{\alpha\cdot k}_{\cdot\beta\cdot q}\Gamma^q_{pm} \tag{4.5}$$

5. The methods developed here can be utilized to study the propagation of waves through a continuous medium /4, 6, 13/ and a flame front /4/, in the theory of plastic flow and fracture /6/, to study surface phenomena /10, 11/ 14/, in dynamic problems of the non-linear

* The operation of the $(\delta/\delta t)$ -derivative (4.3) was introduced in /7/ and the equivalent representation (4.4) in the paper by M.A. Grinfel'd. $(\delta/\delta t)$-derivative and its properties. Dep. v VINITI No.1255-76, Moscow, 1976.

theory of shells, etc.

As an example, we will give expressions for the surface and normal components (5.1) as well as the spatial components (5.2) of the acceleration $\mathbf{j} = (\partial \mathbf{v}/\partial t)_\omega$ of points on a surface moving through three-dimensional space

$$\dot{\gamma}^\alpha = \left(\frac{\partial v^\alpha}{\partial t}\right)_\omega + v^\rho (v^\lambda G^\alpha_{\lambda\rho} - b^\alpha_\rho v_{(n)}) - v_{(n)} \left(b_{\gamma\beta} v^\gamma + \frac{\partial v_{(n)}}{\partial u^\beta}\right) a^{\alpha\beta} \tag{5.1}$$

$$\dot{\gamma}^{\wedge\alpha} = \left(\frac{\partial v^{\wedge\alpha}}{\partial t}\right)_\omega + v^{\wedge\rho} (\nabla^\wedge_\rho v^{\wedge\alpha} - b^{\wedge\alpha}_\rho v_{(n)}) - v_{(n)} \left(b^\wedge_{\gamma\beta} v^{\wedge\gamma} + \frac{\partial v_{(n)}}{\partial \omega^\beta}\right) a^{\wedge\alpha\beta}$$

$$j_{(n)} = \left(\frac{\partial v_{(n)}}{\partial t}\right)_\omega + v^\alpha \left(v^\gamma b_{\gamma\alpha} + \frac{\partial v_{(n)}}{\partial u^\alpha}\right) = \left(\frac{\partial v_{(n)}}{\partial t}\right)_\omega + v^{\wedge\alpha} \left(v^{\wedge\gamma} b^\wedge_{\gamma\alpha} + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\right)$$

$$j^k = \left(\frac{\partial v^k}{\partial t}\right)_\omega + v^i v^p \Gamma^k_{ip} = \left(\frac{\partial v^k}{\partial t}\right)_x + v^p \nabla_p v^k \tag{5.2}$$

It was noted in /11/ that the expressions $j^\alpha = (\partial v^\alpha/\partial t)_\omega$ given in /10, 14/ do not hold in general, and the terms are connected with the change in the local basis at the surface. The latter must be taken into account in the expressions for the components of the acceleration, and (5.1) take this change into account.

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ON THE SURFACE VISCOSITY AT THE BOUNDARY BETWEEN PHASES*

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The equations of motion of the interphase boundary are considered. It is shown that the conditions at the surface separating the phases obtained in /1, 2/ by different methods, are identical. The study of the dynamics of the fluid-fluid interface was initiated by Bussinesq /3/ who postulated a linear relationship between the surface stress tensor $T_{\alpha\beta}$ and the strain rate tensor $S_{\alpha\beta}$, assigning two viscosity coefficients to the surface, the dilatation

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